Analog Design
Kevin Aylward B.Sc
Laplace Transfer Functions

Back to Contents

Abstract

This paper forms an introduction to Laplace Transfer functions as it applies to electronic design. The mathematics is kept to the barest minimum. In practice, for electronic design, the Laplace Transform is used in such a manner that the mathematical basis is usually forgotten. So, for the most part, one can gloss over the mathematics and just use the results, which are trivial to apply in practice.

In reality, there is no need to ever solve the equations. Today this is done in Spice simulation programs. However, to design effectively, one requires a conceptual model of what a transfer function is, and its key features.

Overview

The following are the principle concepts for the introduction of Laplace Transforms.

1 The basic equations of electronics are differential equations with respect to time.

2 Differential equations are harder to solve then algebraic equations.

In short, Laplace Transforms convert differential equations into algebraic equations. These algebraic equations are then solved. The solutions to these equations are then inverted to obtain the time response of an electronic circuit.

In addition, because of the Laplace Transform's relation to the Fourier Transform, the steady state frequency response can also be obtained from the Laplace transformed equations.

Laplace Transform

The Laplace transform is defined by:

$$g(s) = \int_0^\infty f(t) e^{-st} dt$$

If $f(t) = 0$ for $t < 0$ and $s = j\omega$ then $g(j\omega) = \Im(f(t))$ where $\Im$ is the fourier transform

As far as electronics goes, the most important results from the definition, are the following.

Laplace transform of a derivative of function

Consider:

$$h(s)=\int_0^\infty f'(t)e^{-st}dt$$

Integrating by parts.

$$h(s)=\left[f(t)e^{-st}\right]_0^\infty+s\int_0^\infty f(t)e^{-st}dt$$

$$h(s)=0+sg(s)$$

$$h(s)=sg(s)$$

That is the Laplace transform of *the derivative* of a function is the Laplace Transform of the function *multiplied by s*. In general the nth derivative of a function has a Laplace Transform that is multiplied by s^n.

Consider:

$$h(s)=\int_0^\infty f_i(t)e^{-st}dt,\ \text{where } f_i(t) \text{ is the integral of } f(t)$$

Integrating by parts.

$$h(s)=-\frac{1}{s}\left[f_i(t)e^{-st}\right]_0^\infty+\frac{1}{s}\int_0^\infty f(t)e^{-st}dt$$

$$h(s)=-\frac{f_i(0)}{s}e^{-st}+\frac{1}{s}\int_0^\infty f(t)e^{-st}dt$$

For simplicity, assume that $f_i(0) = 0$, then:

$$h(s)=\frac{g(s)}{s}$$

That is the transform of the integral of a function is the transform of the function divided by s.

Note: In most analysis the initial condition of $f_i(0) = 0$ is of no consequence. It can be added in if required.

Application

Consider a series circuit consisting of a voltage source, resistor capacitor and inductor.

$$V_i = IR + L\frac{di}{dt} + \frac{1}{c}\int idt$$

Now multiply the equation by e^{-st} and integrate

$$\int_0^\infty V_i e^{-st} dt = \int_0^\infty I \mathrm{R} e^{-st}\, dt + \int_0^\infty L\frac{di}{dt} e^{-st} dt + \int_0^\infty \frac{1}{c} e^{-st} (\int idt) dt$$

But because of the relations derived above:

$$\overline{V_i}(s) = \overline{I}(s)R + sL\overline{I}(s) + \frac{1}{sc}\overline{I}(s)$$

Were the bars represent the Laplace Transforms of the applied voltage and current.

$$\overline{V_i}(s) = \overline{I}(s)(R + sL + \frac{1}{sc})$$

$$\overline{V_i}(s) = \overline{I}(s)(R + sL + \frac{1}{sc})$$

$$\frac{\overline{V_i}(s)}{\overline{I}(s)} = R + sL + \frac{1}{sc}$$

This is a transfer function. It gives the ration of input voltage to output current in terms of their Laplace transforms. In this particular case, the transfer function is the input impedance of the RLC circuit, in terms of the variable s.

In general, one simple treats inductors and capacitors in the same manner as resistors, by noting that the Laplace impedance of a capacitor is 1/sc and the impedance of an inductor is sl.

Website last modified 30th August 2013

www.kevinaylward.co.uk
