General Relativity For Tellytubbys

Calculus Of Variations

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Overview

This section covers some bits and pieces that may not have been previously covered.

A straight line on a curved surface is *defined* as the shortest distance between two points. This bit is pretty obvious really. Get a map and draw a line from London to New York. Try drawing that line on a globe. Yep, the Concord flies over Greenland, well not anymore it don't, what with the crash and all. So that's the shortest path ain't it, but the flat map shows something way different don't it.

So, given a surface, one needs to find out the equation for these "straight" lines, which are called geodesics. To do this we first need to find out how to find the minimum of an integral. In the particular case of G.R., the integral will be the distance function.

We are going to do a bit of hand waving here, but this reminds me to point out some useful information. For those yanks reading this, you guys have got it completely wrong about Robin Hood. You have this quaint and so naive idea about Europe, that it's a wonder you can tie your own shoelaces. Look, it's a complete misconception that Robin Hood stole from the rich and gave to the poor. In fact, what actually happened was that he stole from the rich and *waved* to the poor.

Euler-Lagrange Differential Equation

Consider the integral

$$\mathrm{I} = \int f(x, y, \dot{y})dx$$

The job is to find the function that minimizes this integral, subject to certain conditions. This is technically described as finding the stationery value of the integral (because that sounds more impressive), which may actually be a local maximum, local minimum or point of inflection, in the following notation.

$$\delta \int f(x, y, y')dx = 0$$

First the answer is:

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y_x}\right) = 0$$

where y_x means the derivative of y wrt x

And no surprise to see that this extends to

$$\frac{\partial f}{\partial y^{\alpha}} - \frac{d}{dx^{\alpha}}(\frac{\partial f}{\partial y_{x^{\alpha}}}) = 0$$

When there are a number of variables, and where the x's are summed over the number of variables.

Now on to the derivation of the above.

Euler-Lagrange Derivation

Consider the function, defined by:

$$\mathrm{u} = y + \alpha.\mathrm{g}(x), \text{such that } \mathrm{g}(a) = \mathrm{g}(b) = 0 \text{ so that at } \alpha = 0 \; \mathrm{u} = \mathrm{y}$$

i.e. vary y about a bit in order to get the best y.

the integral is now

$$\mathrm{I} = \int f(x, \mathrm{u}, \mathrm{u}')dx$$

This will be a turning point for our initial problem if

$$\frac{d\,\mathrm{I}(\alpha)}{d\alpha} = 0 \;\Big|_{\alpha=0}$$

So

$$\frac{d\,\mathrm{I}}{\mathrm{d}\alpha} = \int \frac{d}{d\alpha}\mathrm{f}(x, \mathrm{u}, \mathrm{u}')dx$$

$$\frac{d\,\mathrm{I}(\alpha)}{d\alpha} = \int (\frac{\partial f}{\partial x}\frac{\partial x}{\partial \alpha} + \frac{\partial f}{\partial \mathrm{u}}\frac{\partial \mathrm{u}}{\partial \alpha} + \frac{\partial f}{\partial \mathrm{u}'}\frac{\partial \mathrm{u}'}{\partial \alpha})dx$$

with

$$\frac{\partial \mathrm{u}}{\partial \alpha} = \mathrm{g}(x), \frac{\partial \mathrm{u}'}{\partial \alpha} = \mathrm{g}'(x), \frac{\partial x}{\partial \alpha} = 0$$

So

$$\frac{d\,\mathrm{I}(\alpha)}{d\alpha} = \int (\frac{\partial f}{\partial \mathrm{u}}\mathrm{g}(x) + \frac{\partial f}{\partial \mathrm{u}'}\mathrm{g}'(x))dx$$

Integrating the second term by parts gives

$$\frac{d\,\mathrm{I}(\alpha)}{d\alpha} = \int (\frac{\partial f}{\partial \mathrm{u}}\mathrm{g}(x) - \frac{d}{dx}(\frac{\partial f}{\partial \mathrm{u}'})\mathrm{g}(x))dx + [\frac{\partial f}{\partial \mathrm{u}'}\mathrm{g}(x)]_{x=a}^{x=b}$$

But the 3rd term is zero by construction, so

$$\frac{d\,\mathrm{I}(\alpha)}{d\alpha} = \int (\frac{\partial f}{\partial \mathrm{u}} - \frac{d}{dx}(\frac{\partial f}{\partial \mathrm{u}'}))\mathrm{g}(x)dx$$

But g(x) is arbitrary, so we must have

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y_x}\right) = 0$$

$$\frac{d\,\mathrm{I}(\alpha)}{d\alpha} = 0 \;\Big|_{\alpha=0}$$

After replaying u with y, as required.

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