General Relativity For Tellytubbys

Geodesic Equation

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Overview

This section follows on from the section on Euler-Langrange equations. The task here is to find the geodesic equation that describes straight lines in general.

Geodesic Equation

I do hope you recall from the other pages that, one form of the Euler-Langrange equation is

$$\frac{\partial f}{\partial x^{\alpha}} - \frac{d}{d\lambda}\left(\frac{\partial f}{\partial \dot{x}^{\alpha}}\right) = 0$$

$$I = \int \mathrm{f}(x^{\alpha}, \dot{x}^{\alpha}, \lambda) d\lambda \text{ , where } \dot{x} = \frac{dx}{d\lambda}$$

are the conditions that finds a local minimum, maximums or inflection point of an integral of f.

because that was indeed a waste of brain power, we're going to ignore that just for now, and first derive the geodesic equation directly. This is so we can get a better handle on what's going on from more then one point of view.

Geodesic Equation Method 1

Consider a Tellytubby playing on a slide chute, i.e. undergoing acceleration

$$\mathbf{a} = \frac{d\mathbf{v}}{d\tau}$$

If there are *no* net forces acting on Po (this is the *deeper* meaning bit of G.R.) in order to achieve this acceleration then we have, from Newton's laws

$$m\mathbf{a} = \frac{d\mathbf{v}}{d\tau} = 0$$

In our newly acquired, very impressive tensor notation, this can be written, noting that derivatives go over to covariant derivatives always, as

$$\frac{d\mathbf{v}}{d\tau} = \nabla_{\mathbf{v}} \mathbf{V} \equiv \mathrm{V}^{\alpha}{}_{;\beta} \mathrm{V}^{\beta}$$

because,

$$V^{\alpha}{}_{,\beta}V^{\beta} = \frac{\partial V^{\alpha}}{\partial x^{\beta}}\frac{\partial x^{\beta}}{\partial \tau} = \frac{dV^{\alpha}}{d\tau}$$

and noting the obvious extension to the ";" is required

So, to continue with

$$V^{\alpha}{}_{;\beta}V^{\beta} = 0$$

$$V^{\alpha}{}_{,\beta}V^{\beta} + V^{\beta}V^{\mu}\Gamma^{\alpha}{}_{\mu\beta} = 0$$

guess what index's we swapped now

$$V^{\mu}{}_{,\beta}V^{\beta} + V^{\beta}V^{\alpha}\Gamma^{\mu}{}_{\alpha\beta} = 0$$

but we have $V^{\alpha} \equiv \frac{dx^{\alpha}}{d\tau}$

and so the first term can be written as

$$V^{\mu}{}_{,\beta}V^{\beta} = \frac{\partial V^{\mu}}{\partial x^{\beta}}\frac{\partial x^{\mu}}{\partial \tau} = \frac{dV^{\mu}}{d\tau} = \frac{d^2 x^{\mu}}{d\tau^2} \text{ by our wonderfull chain rule}$$

and subbing in again to all terms gets us

$$\frac{d^2 x^{\mu}}{d\tau^2} + \Gamma^{\mu}{}_{\alpha\beta}\frac{dx^{\alpha}}{d\tau}\frac{dx^{\beta}}{d\tau} = 0$$

Which is the geodesic equation that we are after.

So, this gives one a bit of a feel, one hopes, of what is happening dude

Geodesic Equation Method 2

Now to do the difficult bit and show how things all tie up with the variational principle

Consider the path that light takes

$c = \frac{ds}{d\tau}$, where s is the distance traveled and τ is defined as the proper time

so that, using our prior result for distance, one can write

$$d\tau^2 = \frac{1}{c^2}ds^2 = \frac{1}{c^2}g_{\beta\gamma}dx^{\beta}dx^{\gamma}$$

To make the sums all work out, an "affine parameter " for the time is introduced. This is simply to get rid of all those dx's, bloody annoyance that they are.

So, let $\tau = \tau(\lambda)$ then $d\tau = \frac{d\tau}{d\lambda}d\tau$

and dividing out by dλ in our distance formula above gives, well after taking the square root and all

$$\frac{d\tau}{d\lambda} = \frac{1}{c}\frac{ds}{d\lambda} = \frac{1}{c}\left[g_{\beta\gamma}\frac{dx^\beta}{d\lambda}\frac{dx^\gamma}{d\lambda} \right]^{\frac{1}{2}}$$

Hence:

$$d\tau = \frac{1}{c}\left[g_{\beta\gamma}\frac{dx^\beta}{d\lambda}\frac{dx^\gamma}{d\lambda} \right]^{\frac{1}{2}} d\lambda$$

or finding the total time

$$\tau = \int \frac{1}{c}\left[g_{\beta\gamma}(x^\alpha)\frac{dx^\beta}{d\lambda}\frac{dx^\gamma}{d\lambda} \right]^{\frac{1}{2}} d\lambda$$

So, now the job is to minimize this integral, ~~Laa Laa~~ oops, I mean ala this is the celebrated least action integral for our geodesic.

When I was ~~plagiarizing~~ researching for this project on the web I found one or two derivations of this result. However, they were all rather more complicated because it is obvious that whatever locally minimizes $f^{1/2}$, will also locally minimize plain old f as well, so we'll drop the square root complication and just consider:

$$\tau = \int g_{\beta\gamma}(x^\alpha)\frac{dx^\beta}{d\lambda}\frac{dx^\gamma}{d\lambda} d\lambda$$

$$\frac{\partial f}{\partial x^\alpha} - \frac{d}{d\lambda}\left(\frac{\partial f}{\partial \dot{x}^\alpha}\right) = 0$$

First term, and note we have dropped c because we are equating to 0

$$\frac{\partial f}{\partial x^\alpha} = g_{\beta\gamma,\alpha}\frac{dx^\beta}{d\lambda}\frac{dx^\gamma}{d\lambda}$$

Second term

$$\frac{\partial f}{\partial \dot{x}^\alpha} = g_{\beta\gamma}\frac{\partial}{\partial \dot{x}^\alpha}\left(\frac{dx^\beta}{d\lambda}\frac{dx^\gamma}{d\lambda}\right)$$

$$\frac{\partial f}{\partial \dot{x}^\alpha} = g_{\beta\gamma}\frac{\partial}{\partial \dot{x}^\alpha}(\dot{x}^\beta \dot{x}^\gamma)$$

where I've changed the notation to make it a bit clearer what's going on. So mentally ignore the dots on the x's when doing the sums. I have filled in *all* the steps because they were not done in the derivation where I copied the outline of this from. What these poor excuse's for Tellytubby professors don't realize is that, precisely because the reader is going through these elementary deviations, it inherently implies that the punter *is not* familiar with these sorts of calculations, and so more guidance is needed. e.g. Note how the delta swap's index's.

$$\frac{\partial f}{\partial \dot{x}^\alpha} = g_{\beta\gamma}\left[\frac{\partial \dot{x}^\beta}{\partial \dot{x}^\alpha}\dot{x}^\gamma + \dot{x}^\beta \frac{\partial \dot{x}^\gamma}{\partial \dot{x}^\alpha} \right]$$

$$\frac{\partial f}{\partial \dot{x}^{\alpha}} = g_{\beta\gamma}\left[\frac{\partial \dot{x}^{\beta}}{\partial \dot{x}^{\gamma}}\frac{\partial \dot{x}^{\gamma}}{\partial \dot{x}^{\alpha}}\dot{x}^{\gamma} + \dot{x}^{\beta}\delta^{\gamma}_{\alpha}\right]$$

$$\frac{\partial f}{\partial \dot{x}^{\alpha}} = g_{\beta\gamma}\left[\delta^{\beta}_{\gamma}\delta^{\gamma}_{\alpha}\dot{x}^{\gamma} + \dot{x}^{\beta}\delta^{\gamma}_{\alpha}\right]$$

$$\frac{\partial f}{\partial \dot{x}^{\alpha}} = g_{\beta\gamma}\delta^{\gamma}_{\alpha}\left[\delta^{\beta}_{\gamma}\dot{x}^{\gamma} + \dot{x}^{\beta}\right]$$

$$\frac{\partial f}{\partial \dot{x}^{\alpha}} = g_{\beta\gamma}\delta^{\gamma}_{\alpha}\left[\dot{x}^{\beta} + \dot{x}^{\beta}\right]$$

$$\frac{\partial f}{\partial \dot{x}^{\alpha}} = 2.g_{\beta\alpha}\dot{x}^{\beta}$$

We now have then

$$g_{\beta\gamma,\alpha}\frac{dx^{\beta}}{d\lambda}\frac{dx^{\gamma}}{d\lambda} - \frac{d}{d\lambda}\left[2.g_{\alpha\beta}\frac{dx^{\beta}}{d\lambda}\right] = 0$$

$$g_{\beta\gamma,\alpha}\frac{dx^{\beta}}{d\lambda}\frac{dx^{\gamma}}{d\lambda} - 2.g_{\alpha\beta}\frac{d}{d\lambda}(\frac{dx^{\beta}}{d\lambda}) - 2.\frac{d}{d\lambda}(g_{\alpha\beta}).\frac{dx^{\beta}}{d\lambda} = 0$$

$$g_{\beta\gamma,\alpha}\frac{dx^{\beta}}{d\lambda}\frac{dx^{\gamma}}{d\lambda} - 2.g_{\alpha\beta}\frac{d^2x^{\beta}}{d\lambda^2} - 2.\frac{d}{dx^{\gamma}}(g_{\alpha\beta})\frac{dx^{\gamma}}{d\lambda}\frac{dx^{\beta}}{d\lambda} = 0$$

$$g_{\beta\gamma,\alpha}\frac{dx^{\beta}}{d\lambda}\frac{dx^{\gamma}}{d\lambda} - 2.g_{\alpha\beta}\frac{d^2x^{\beta}}{d\lambda^2} - 2.g_{\alpha\beta,\gamma}\frac{dx^{\gamma}}{d\lambda}\frac{dx^{\beta}}{d\lambda} = 0$$

$$g^{\mu\alpha}g_{\beta\gamma,\alpha}\frac{dx^{\beta}}{d\lambda}\frac{dx^{\gamma}}{d\lambda} - 2.g^{\mu\alpha}g_{\alpha\beta}\frac{d^2x^{\beta}}{d\lambda^2} - 2.g^{\mu\alpha}g_{\alpha\beta,\gamma}\frac{dx^{\gamma}}{d\lambda}\frac{dx^{\beta}}{d\lambda} = 0$$

$$2.\frac{d^2x^{\mu}}{d\lambda^2} - g^{\mu\alpha}g_{\beta\gamma,\alpha}\frac{dx^{\beta}}{d\lambda}\frac{dx^{\gamma}}{d\lambda} + 2.g^{\mu\alpha}g_{\alpha\beta,\gamma}\frac{dx^{\gamma}}{d\lambda}\frac{dx^{\beta}}{d\lambda} = 0$$

$$2.\frac{d^2x^{\mu}}{d\lambda^2} - g^{\mu\alpha}g_{\beta\gamma,\alpha}\frac{dx^{\beta}}{d\lambda}\frac{dx^{\gamma}}{d\lambda} + g^{\mu\alpha}g_{\alpha\beta,\gamma}\frac{dx^{\gamma}}{d\lambda}\frac{dx^{\beta}}{d\lambda} + g^{\mu\alpha}g_{\alpha\beta,\gamma}\frac{dx^{\gamma}}{d\lambda}\frac{dx^{\beta}}{d\lambda} = 0$$

$$\frac{d^2x^{\mu}}{d\lambda^2} + \frac{g^{\mu\alpha}}{2}\left[g_{\alpha\beta,\gamma}\frac{dx^{\gamma}}{d\lambda}\frac{dx^{\beta}}{d\lambda} + g_{\alpha\beta,\gamma}\frac{dx^{\gamma}}{d\lambda}\frac{dx^{\beta}}{d\lambda} - g_{\beta\gamma,\alpha}\frac{dx^{\beta}}{d\lambda}\frac{dx^{\gamma}}{d\lambda}\right] = 0$$

Ahmm, getting close, seems familiar? Swap one more time

$$\frac{d^2x^{\mu}}{d\lambda^2} + \frac{g^{\mu\alpha}}{2}\left[g_{\alpha\beta,\gamma}\frac{dx^{\gamma}}{d\lambda}\frac{dx^{\beta}}{d\lambda} + g_{\alpha\gamma,\beta}\frac{dx^{\beta}}{d\lambda}\frac{dx^{\gamma}}{d\lambda} - g_{\beta\gamma,\alpha}\frac{dx^{\beta}}{d\lambda}\frac{dx^{\gamma}}{d\lambda}\right] = 0$$

$$\frac{d^2x^\mu}{d\lambda^2}+\frac{g^{\mu\alpha}}{2}\left[g_{\alpha\beta,\gamma}+g_{\alpha\gamma,\beta}-g_{\beta\gamma,\alpha}\right]\frac{dx^\beta}{d\lambda}\frac{dx^\gamma}{d\lambda}=0$$

which, by referring to our Christoffel page, is

$$\frac{d^2x^\mu}{d\lambda^2}+\Gamma^\mu{}_{\beta\gamma}\frac{dx^\beta}{d\lambda}\frac{dx^\gamma}{d\lambda}=0$$

and, obviously, we can let tau = lambda

amazing, ain't it. How different methods give the same answer.

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http://www.kevinaylward.co.uk/gr/index.html

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