General Relativity For Tellytubbys

The Covariant Derivative

Sir Kevin Aylward B.Sc., Warden of the Kings Ale

Back to the Contents section

The approach presented here is one of the most direct routes possible. I plagiarized it from a number of sources and added my bit of finesse to it i.e. no introduction to all that superfluous mumbo jumbo that disappears without actually doing anything, but confuses you like no bodies business. Most steps are shown in complete detail, cos I remember I was totally lost when I first learned this stuff.

So, no beating about the bush, onward with a quick derivation of the covariant derivative.

Section 1

Consider a vector or tensor of rank 1, with components

$$V^{\alpha}$$

or in full notation

$$\mathbf{V} = V^{\alpha}\mathbf{e}_{\alpha}$$

The covariant derivative is defined by deriving the second order tensor obtained by

$$\frac{\partial \mathbf{V}}{\partial x^{\beta}} = (V^{\alpha}\mathbf{e}_{\alpha})_{;\beta} = \frac{\partial V^{\alpha}}{\partial x^{\beta}}\mathbf{e}_{\alpha} + V^{\alpha}\frac{\partial \mathbf{e}_{\alpha}}{\partial x^{\beta}}$$

No mystery at all here, we just have to account for the fact that the basis vectors are not constant by using the usual differentiation of the product rule. Note the ";" to indicate the covariant derivative.

The last term containing the derivative of the basis vector can clearly be expressed as a sum of the basis vectors themselves. This will be written as:

$$\mathbf{e}_{\alpha,\beta} = \frac{\partial \mathbf{e}_{\alpha}}{\partial x^{\beta}} = \Gamma^{\mu}{}_{\alpha\beta}\mathbf{e}_{\mu}$$

Where the big funny R shape is to be determined, and is called the Christoffel symbol of the 2nd kind, and not to be confused with Close Encounters of the Third Kind, which was crap, and almost as bad as ET it was.

Also note the new introduction of "," to mean ordinary partial derivative.

So now the covariant derivative can be written as:

$$\frac{\partial \mathbf{V}}{\partial x_{\beta}} = \frac{\partial V^{\alpha}}{\partial x_{\beta}}\mathbf{e}_{\alpha} + V^{\alpha}\Gamma^{\mu}{}_{\alpha\beta}\,\mathbf{e}_{\mu}$$

Letting $\mu -> \alpha$ and $\alpha -> \mu$ in the second term gives

$$\frac{\partial \mathbf{V}}{\partial x_\beta} = \frac{\partial V^\alpha}{\partial x_\beta}\mathbf{e}_\alpha + V^\mu \Gamma^\alpha{}_{\mu\beta}\,\mathbf{e}_\alpha$$

$$\frac{\partial \mathbf{V}}{\partial x_\beta} = (\frac{\partial V^\alpha}{\partial x_\beta} + V^\mu \Gamma^\alpha{}_{\mu\beta}\,)\mathbf{e}_\alpha$$

which means that the covariant derivative of the vector, specified only by its components, can now be expressed as

$$V^\alpha{}_{;\beta} = V^\alpha{}_{,\beta} + V^\mu \Gamma^\alpha{}_{\mu\beta}$$

Which is indeed a tensor, but we certainly don't care one iota about proving that it is a tensor, some other fool can do that.

Now lets consider a second order tensor

$$\mathbf{T} = T^{\alpha\gamma}\mathbf{e}_\alpha \otimes \mathbf{e}_\gamma$$

Then calculating its covariant derivative by differentiating by parts, and using the above results, gives

$$\mathbf{T}_{;\rho} = T^{\alpha\beta}{}_{,\rho}\mathbf{e}_\alpha \otimes \mathbf{e}_\gamma + T^{\alpha\gamma}\mathbf{e}_{\alpha,\rho} \otimes \mathbf{e}_\gamma T^{\alpha\gamma}\mathbf{e}_\alpha \otimes \mathbf{e}_{\gamma,\rho}$$

or, after expanding and collecting up terms as we did above

$$T^{\alpha\beta}{}_{;\rho} = T^{\alpha\beta}{}_{,\rho} + T^{\mu\beta}\Gamma^\alpha{}_{\mu\rho} + T^{\alpha\mu}\Gamma^\beta{}_{\mu\rho}$$

And if your so inclined, you can go and derive the covariant derivative for a downstairs index as

$$T^\alpha{}_{\beta;\rho} = T^\alpha{}_{\beta,\rho} + T^\mu{}_\beta \Gamma^\alpha{}_{\mu\rho} - T^\alpha{}_\mu \Gamma^\mu{}_{\beta\rho}$$

Section 2

Next task my Tellytubbys, is to derive the Christoffel symbols so that we can actually do something.

To start off, the Symmetry of $\Gamma^\alpha{}_{\mu\beta}$ is first shown

By definition $\mathbf{e}_\alpha = \frac{\partial \mathbf{R}}{\partial x^\alpha}$, back to the other pages for refresher if you've forgotten this

So $\mathbf{e}_{\alpha,\beta} = \frac{\partial}{\partial x^\beta}\frac{\partial \mathbf{R}}{\partial x^\alpha} = \frac{\partial}{\partial x^\alpha}\frac{\partial \mathbf{R}}{\partial x^\beta} = \mathbf{e}_{\beta,\alpha}$

Thus $\mathbf{e}_{\alpha,\beta}$ is symmetrical wrt α and β

From our previous result above,

$$\mathbf{e}_{\alpha,\beta} = \Gamma^\mu{}_{\alpha\beta}\mathbf{e}_\mu$$

then

$$\mathbf{e}_{\beta,\alpha} = \Gamma^\mu{}_{\beta\alpha}\mathbf{e}_\mu = \mathbf{e}_{\alpha,\beta} = \Gamma^\mu{}_{\alpha\beta}\mathbf{e}_\mu$$

$$\Gamma^{\mu}{}_{\beta\alpha} = \Gamma^{\mu}{}_{\alpha\beta}$$

is also symmetrical wrt alpha and beta

Laa Laa now writes from the above

$$\mathbf{e}_{\alpha,\beta}.\mathbf{e}_{\lambda} = \Gamma^{\mu}{}_{\alpha\beta}\mathbf{e}_{\mu}.\mathbf{e}_{\lambda}$$

or

$$\Gamma^{\mu}{}_{\alpha\beta}\mathbf{e}_{\mu}.\mathbf{e}_{\lambda} = \mathbf{e}_{\alpha,\beta}.\mathbf{e}_{\lambda}$$

$$g_{\mu\lambda}\Gamma^{\mu}{}_{\alpha\beta} = \mathbf{e}_{\alpha,\beta}.\mathbf{e}_{\lambda}$$

$$g^{\lambda\nu}g_{\mu\lambda}\Gamma^{\mu}{}_{\alpha\beta} = g^{\lambda\nu}[\mathbf{e}_{\alpha,\beta}.\mathbf{e}_{\lambda}]$$

$$g^{\nu}{}_{\mu}\Gamma^{\mu}{}_{\alpha\beta} = g^{\lambda\nu}[\mathbf{e}_{\alpha,\beta}.\mathbf{e}_{\lambda}]$$

$$\Gamma^{\nu}{}_{\alpha\beta} = g^{\lambda\nu}[\mathbf{e}_{\alpha,\beta}.\mathbf{e}_{\lambda}]$$

and due to the symmetry found above, we can also write

$$\Gamma^{\nu}{}_{\alpha\beta} = g^{\lambda\nu}[\mathbf{e}_{\beta,\alpha}.\mathbf{e}_{\lambda}]$$

So adding these last two gives

$$\Gamma^{\nu}{}_{\alpha\beta} = \frac{1}{2}g^{\nu\lambda}[\mathbf{e}_{\alpha,\beta}.\mathbf{e}_{\lambda} + \mathbf{e}_{\beta,\alpha}.\mathbf{e}_{\lambda}]$$

So, by re-differentiating, the following line can immediately be seen to be correct, don't you just love these ones, ah.

$$\Gamma^{\nu}{}_{\alpha\beta} = \frac{1}{2}g^{\nu\lambda}[(\mathbf{e}_{\alpha}.\mathbf{e}_{\lambda})_{,\beta} + (\mathbf{e}_{\beta}.\mathbf{e}_{\lambda})_{,\alpha} - \mathbf{e}_{\alpha}.\mathbf{e}_{\lambda,\beta} - \mathbf{e}_{\beta}.\mathbf{e}_{\lambda,\alpha}]$$

$$\Gamma^{\nu}{}_{\alpha\beta} = \frac{1}{2}g^{\nu\lambda}[g_{\alpha\lambda,\beta} + g_{\beta\lambda,\alpha} - \mathbf{e}_{\alpha}.\mathbf{e}_{\lambda,\beta} - \mathbf{e}_{\beta}.\mathbf{e}_{\lambda,\alpha}]$$

$$\Gamma^{\nu}{}_{\alpha\beta} = \frac{1}{2}g^{\nu\lambda}[g_{\alpha\lambda,\beta} + g_{\beta\lambda,\alpha} - \mathbf{e}_{\alpha}.\mathbf{e}_{\beta,\lambda} - \mathbf{e}_{\beta}.\mathbf{e}_{\alpha,\lambda}]\text{, using the result } \mathbf{e}_{\lambda,\alpha} = \mathbf{e}_{\alpha,\lambda} \text{ again}$$

$$\Gamma^{\nu}{}_{\alpha\beta} = \frac{1}{2}g^{\nu\lambda}[g_{\alpha\lambda,\beta} + g_{\beta\lambda,\alpha} - (\mathbf{e}_{\alpha}.\mathbf{e}_{\beta})_{,\lambda}]$$

$$\Gamma^{\nu}{}_{\alpha\beta} = \frac{1}{2}g^{\nu\lambda}[g_{\alpha\lambda,\beta} + g_{\beta\lambda,\alpha} - g_{\alpha\beta,\lambda}]$$
